EECS 127 Lect 2

- Midterm Oct 12 7-9 pm
- Ranade OH:

  Tu 565 Cory after lec

  Thu outside after lec

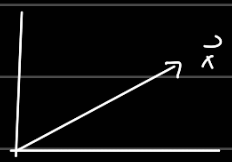
Linear Algebra Bootcamp

- Vectors, Norms
- Gram-Schmidt
- Fundamental Theorem of Linear Algebra

Vector : $\vec{x} \in \mathbb{R}^n$

- 2-norm / Euclidean norm

$$\|\vec{x}\|_2 = \sqrt{\sum_{i=1}^{n}}$$



- **Norm** : Vector Space $\mathcal{X}$

  A function $f : \mathcal{X} \longrightarrow \mathbb{R}$ is a norm if

  1) $\|\vec{x}\| \geq 0 \quad \forall \vec{x} \in \mathcal{X}$; $\|\vec{x}\| = 0$ iff $\vec{x} = 0$
  2) Triangle inequality : $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$
     $\forall \vec{x}, \vec{y} \in \mathcal{X}$
  3) Scalar mult.
     $\|\alpha \vec{x}\| = \alpha \|\vec{x}\|$
     $\forall \alpha \in \mathbb{R}, \forall \vec{x} \in \mathcal{X}$

Example : ℓ_p norm

$$\|\vec{x}\|_p \doteq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \; ; \; 1 \leq p < \infty$$

Choose $p = 2 \rightarrow$ Euclidean norm

Choose $p = 1$

$$\|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

Choose $p = \infty$

$$\lim_{p \to \infty} \|\vec{x}\|_p = \max |x_i| \text{ of the vector}$$

$\max$ ~ largest element of the vector

Cauchy-Schwartz Inequality

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = \|\vec{x}\|_2 \|\vec{y}\|_2 \cos(\theta)$$

↳ θ : ∡ btwn x & y

$$\langle \vec{x}, \vec{y} \rangle \leq \|\vec{x}\|_2 \|\vec{y}\|_2$$

$$|\vec{x}^T \vec{y}| \leq \|\vec{x}\|_2 \|\vec{y}\|_2$$

→ extending C-S wrt l_p-norm: Hölder's Inequality

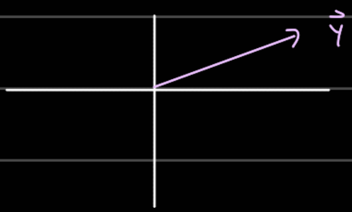
$p, q \geq 1$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$

$$|\vec{x}^T \vec{x}| \leq \sum_{i=1}^{n} |x_i y_i| \leq \|\vec{x}\|_p \|\vec{y}\|_p$$

First optimization problem:

$$\max \vec{x}^T \vec{y} \quad \text{(maximizing an inner product)}$$

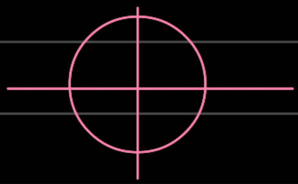
$\|\vec{x}\|_p \leq 1$, $\vec{y} \in \mathbb{R}^n$ fixed & given



$\vec{y} \in \mathbb{R}^2$

$p = 2$ case

$\|\vec{x}\|_2 \leq 1$

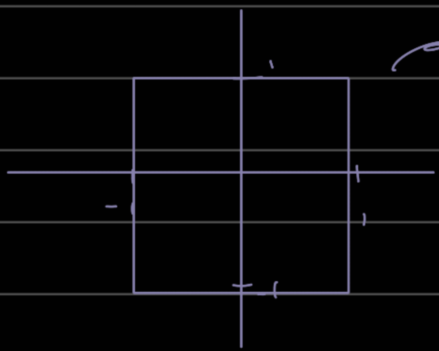


↳ choose unit vector in direction of $\vec{y}$

↓

$$\vec{x}^* = \frac{\vec{y}}{\|\vec{y}\|_2}$$

$p = \infty$ case



bc we're working w/ max val. $= 1$

$\vec{x}^T \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$

→ Choose $\vec{x}$ to maximize $\vec{y}$.

→ $-1 \leq x_i \leq 1$

↳ $x_i = \text{sgn}(y_i)$ ← the sign of

$\vec{x} = \text{sgn}(\vec{y})$

↳ if $y_1 = -10$, then $x_1 = -1$

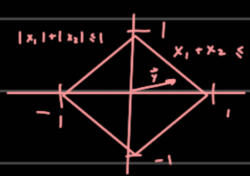
if $y_1 = 10$, $x_1 = 1$

$$\max_{\|\vec{x}\|_\infty \leq 1} \vec{x}^T \vec{y} = \sum_{i=1}^{n} |y_i|$$

$$= \|\vec{y}\|_1$$

Choose $p = 1$

$\max \ \vec{x}^T \vec{y} \quad \text{s.t.} \ \|\vec{x}\| \leq 1$



$\vec{x}^T \vec{y} = x_1 y_1 + \cdots + x_n y_n$

$\vec{x}^T \vec{y} \leq |\vec{x}^T \vec{y}| = \left|\sum_{i=1}^{n} x_i y_i\right|$

Apply triangle inequality:

$$\left|\sum_{i=1}^{n} x_i y_i\right| \leq \sum_{i=1}^{n} |x_i y_i| = \sum_{i=1}^{n} |x_i||y_i|$$

$\downarrow$

$|y_{max}|$ = largest absolute value

$\downarrow$

$$\sum_{i=1}^{n} |x_i||y_i| \leq \sum_{i=1}^{n} |x_i||y_{max}|$$

$$* = |y_{max}|\left(\sum |x_i|\right) \leq |y_{max}| = \|\vec{y}_\infty\|$$

To get max:

$y_1 \quad y_2 \quad \cdots \quad y_{max} \quad \cdots \quad y_n$

$\vec{x}: \quad 0 \quad 0 \quad \cdots \quad sgn(y_{max}) \cdots 0$

Gram-Schmidt (Vector Orthonormalization) / QR-Decomposition

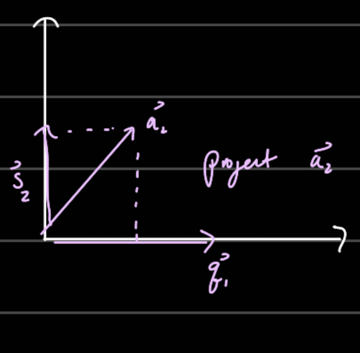
χ : $\{ \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n \}$ Basis

↳ G-S is a method to get an orthonormal basis for the vector space

Gram-Schmidt Steps

① $\vec{q}_1 = \dfrac{\vec{a}_1}{\|\vec{a}_1\|_2}$ (normalize $\vec{a}_1$ - don't need to check orthogonality bc it's the first vector)

②



project $\vec{a}_2$ onto $\vec{q}_1$ } $= \vec{q}_1 \langle \vec{a}_2, \vec{q}_1 \rangle$

↳ only true bc $\vec{q}_1$ is normalized

find the residual $\vec{s}_2$ } $\vec{s}_2 = \vec{a}_2 - \vec{q}_1 \langle \vec{a}_2, \vec{q}_1 \rangle$

$\rightarrow$ now normalize to get $\vec{q}_2$

$$\vec{q}_2 = \frac{\vec{s}_2}{\|\vec{s}_2\|}$$

③ $\vec{q}_3 \rightarrow$ span $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$ = span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

$$\vec{q}_3 \perp \vec{q}_1, \vec{q}_2$$

$$\vec{s}_3 = \vec{a}_3 - \langle \vec{a}_3, \vec{q}_1 \rangle \vec{q}_1 - \langle \vec{a}_3, \vec{q}_2 \rangle \vec{q}_2$$

$$\vec{q}_3 = \frac{\vec{s}_3}{\|\vec{s}_3\|}$$

QR - Decomposition:

$$A = Q \cdot R$$

$$\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_n \end{bmatrix} \underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & & \vdots \\ 0 & & \cdots & \\ & & & r_{nn} \end{bmatrix}}_{\text{upper triangular}}$$